Problems posed in the
2018 Miklós Schweitzer Memorial Competition in Mathematics

26 October – 5 November 2018

1. Let $S \subset \mathbb{R}$ be a closed set and $f : \mathbb{R}^{2n} \to \mathbb{R}$ be a continuous function. Let us define a graph G as follows. Let x be a vertex of G iff $x \in \mathbb{R}^n$ and $f(x,x) \notin S$. We connect the vertices x and y by an edge in G iff $f(x,y) \in S$ or $f(y,x) \in S$. Show that the chromatic number of G is countable.

2. A family $\mathcal{F}$ of sets is called *really neat* if for any $A, B \in \mathcal{F}$ there is a set $C \in \mathcal{F}$ such that $A \cup B = A \cup C = B \cup C$. Let

$$f(n) = \min \left\{ \max_{A \in \mathcal{F}} |A| \ : \ \mathcal{F} \text{ is really neat and } |\cup \mathcal{F}| = n \right\}.$$

Prove that the sequence $f(n)/n$ converges and find its limit.

3. We call an $n \times n$ matrix *well groomed* if it only contains the elements 0 and 1, and it does not contain the submatrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Show that there exists a constant $c > 0$ such that every well groomed, $n \times n$ matrix contains a submatrix of size at least $cn \times cn$ such that all of the elements of the submatrix are equal. (A well groomed matrix may contain the submatrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.)

4. Let P be a finite set of points in the plane. Assume that the distance between any two points is an integer. Prove that P can be colored by three colors such that the distance between any two points of the same color is an even number.

5. For every positive integer n, define

$$f(n) = \sum_{p|n} p^{k_p},$$

where the sum is taken over all positive prime divisors p of n, and k_p is the unique integer satisfying

$$p^{k_p} \leq n < p^{k_p+1}.$$

Find

$$\limsup_{n\to\infty} \frac{f(n)\log\log n}{n \log n}.$$

6. Prove that if a is an integer and d is a positive divisor of the number $a^4 + a^3 + 2a^2 - 4a + 3$, then d is a fourth power modulo 13.

7. Describe all functions $f : \{0,1\}^n \to \{0,1\}$ which satisfy the equation

$$\begin{aligned} &f(f(a_{11}, a_{12}, \ldots, a_{1n}), f(a_{21}, a_{22}, \ldots, a_{2n}), \ldots, f(a_{n1}, a_{n2}, \ldots, a_{nn})) \\ &\qquad = f(f(a_{11}, a_{21}, \ldots, a_{n1}), f(a_{12}, a_{22}, \ldots, a_{n2}), \ldots, f(a_{1n}, a_{2n}, \ldots, a_{nn})) \end{aligned}$$

for arbitrary $a_{ij} \in \{0,1\}$ $(i,j = 1,2,\ldots,n)$.

8. Does there exist a piecewise linear, continuous, surjective mapping $f : [0,1] \to [0,1]$ such that $f(0) = f(1) = 0$, and for all positive integers n,

$$2.0001^{(n-10)} < P_n(f) < 2.9999^{(n+10)}$$

holds, where $P_n(f)$ is the number of points x such that $\underbrace{f(\ldots f}_{n}(x)\ldots) = x$?

9. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function, and suppose that the sequence $f^{(n)}$ of derivatives converges pointwise. Prove that $f^{(n)}(z) \to Ce^z$ pointwise for a suitable complex number C.

10. In 3-dimensional hyperbolic space, we are given a plane P and four distinct straight lines: the lines a_1 and a_2 are perpendicular to P; while the lines r_1 and r_2 do not intersect P, and their distances from P are equal. Denote by S_i the surface of revolution obtained by rotating r_i around a_i. Show that the common points of S_1 and S_2 can be covered by two planes.

11. We call an m-dimensional smooth manifold *parallelizable* if it admits m smooth tangent vector fields that are linearly independent at all points. Show that if M is a closed orientable $2n$-dimensional smooth manifold of Euler characteristic 0 that has an immersion into a parallelizable smooth $(2n+1)$-dimensional manifold N, then M is itself parallelizable.

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The Schweitzer Miklós Competition is open for all students who currently study in Hungary and also to Hungarian citizens studying abroad, who do not hold an MSc (or equivalent) degree in mathematics, computer science or related fields obtained in 2017 or earlier. (Those who graduated in 2018 are eligible to participate.)

Please submit solutions to the problems on separate sheets by 12:00 (noon) CET, 5 November 2018,

- *in person* to the Secretariat of ELTE Mathematics Department (1117 Budapest, Pázmány Péter sétány 1/C, room 3-510)
- or *by registered mail* to:

ELTE Matematikai Intézet, Keleti Tamás
1117 Budapest, Pázmány Péter sétány 1/C.

- or *by email in PDF format* to kiss.viktor@renyi.mta.hu.

Please write your name on each sheet and your affiliation and email on one sheet.

Problems are to be solved individually, cooperation of any form is strictly forbidden.

Please do not pose or discuss any of these problems at any online forum before the end of the competition, and let us know immediately if you notice that this happens.