Problems posed in the 2024 Miklós Schweitzer Memorial Competition in Mathematics

25 October 2024 – 4 November 2024

1. Let $G = (S, T; E)$ be a finite bipartite graph with a perfect matching. Show that there exists an injective weight function $w\colon E \to \mathbb{R}$ on the edges with the following properties:

(1) If e_s denotes the edge at vertex $s \in S$ with minimal weight, then $\{e_s \mid s \in S\}$ is a perfect matching in G.

(2) If e_t denotes the edge at vertex $t \in T$ with maximal weight, then $\{e_t \mid t \in T\}$ is a perfect matching in G.

2. Does there exist a nowhere dense, nonempty compact set $C \subset [0, 1]$ such that

$$\liminf_{h\to 0+} \frac{\lambda\big(C \cap (x, x+h)\big)}{h} > 0 \qquad \text{or} \qquad \liminf_{h\to 0+} \frac{\lambda\big(C \cap (x-h, x)\big)}{h} > 0$$

holds for every point $x \in C$, where $\lambda(A)$ denotes the Lebesgue measure of A?

3. Do there exist nowhere differentiable continuous functions $f, g\colon \mathbb{R} \to \mathbb{R}$ such that $f \circ g$ is differentiable?

4. Let π be a given permutation of the set $\{1, 2, \ldots, n\}$. Find the smallest possible value of $\sum\limits_{i=1}^{n} |\pi(i) - \sigma(i)|$ if the permutation σ may be chosen to be any cycle of length n. Write the answer as a function of the number and lengths of the cycles (including those of length 1) appearing in the decomposition of π as a product of disjoint cycles.

5. Let X be a regular topological space and let S be a countably compact dense subspace in X. (The countably compact property means that every infinite subset of S has an accumulation point in S.) Show that S is also G_δ-dense in X, i.e. S intersects all nonempty G_δ sets.

6. During heat diffusion, we say that the evolution of temperature at a point $x \in \mathbb{R}^n$ is *astonishing* if it changes monotonicity infinitely many times. Can it happen that the temperature evolves astonishingly at every point $x \in \mathbb{R}^n$? More precisely, does there exist a nonnegative $u \in C^2((0, +\infty) \times \mathbb{R}^n)$ solving the heat equation $\partial_t u - \Delta u = 0$, such that $u(t, x) \to 0$ for every t as $|x| \to \infty$, and for every $x \in \mathbb{R}^n$ there exists a monotone sequence (t_k) of positive numbers such that $(-1)^k \partial_t u(t_k, x) > 0$ for every k?

7. Is it true that if a subgroup $G \leq \mathrm{Sym}(\mathbb{N})$ is n-transitive for every positive integer n, then every group automorphism of G extends to a group automorphism of $\mathrm{Sym}(\mathbb{N})$?

8. Prove that for any finite bipartite planar graph, a circle can be assigned to each vertex so that all circles are coplanar, the circles assigned to any two adjacent vertices are tangent to one another, while the circles assigned to any two distinct, non-adjacent vertices intersect in two points.

9. Let $q > 1$ be a power of 2. Let $f\colon \mathbb{F}_{q^2} \to \mathbb{F}_{q^2}$ be an affine map over $\mathbb{F}_2$. Show that the equation $f(x) = x^{q+1}$ has at most $2q - 1$ solutions.

10. Let $A > 0$ and $B = \big(3 + 2\sqrt{2}\big)\, A$. Show that in the finite sequence $a_k = \big\lfloor k/\sqrt{2} \big\rfloor$ ($k \in (A, B) \cap \mathbb{Z}$) the numbers of even, respectively odd terms differ by at most 2.

11. An urn initially contains one red and one blue ball. In each step, we choose a uniform random ball from the urn. If it is red, then another red ball and another blue ball are placed in the urn. And when we choose a blue ball for the k-th time, we put a blue ball and $2k + 1$ red balls in the urn. (The chosen balls are not removed, they remain in the urn.)

Let G_n denote the number of balls in the urn after n steps. Prove that there exist constants $0 < c, \alpha < \infty$ such that $\frac{G_n}{n^\alpha} \to c$ almost surely.

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The Schweitzer Miklós Competition is open for all students who currently study in Hungary or are Hungarian citizens studying abroad, who do not hold an MSc (or equivalent) degree in mathematics, computer science or related fields obtained in 2023 or earlier. (Those who graduated in 2024 are eligible to participate.)

This year, competitors will have 10 days to solve the problems. Solutions may be submitted by **12:00 PM (noon) CET, 4 November 2024**, via email to the following address: schweitzer.miklos@gmail.com. Please include your full name, affiliation in Hungary, home address and email address when submitting. Solutions should be in **PDF format** (in case of scanned documents, please ensure the legibility of the image), preferably sent in a single letter but **in separate files** for each problem, and written in Hungarian. In exceptional cases, following prior approval, we can accept solutions written in English as well. Any papers submitted after the deadline or that otherwise do not satisfy the above criteria will not be considered by the committee.

Participants may submit solutions to any subset of the problems. Prizes will be handed out to the best performing students by the János Bolyai Mathematical Society. Solving at least three problems correctly will automatically result in receiving a prize or certificate.

Problems are to be solved individually, cooperation of any form is strictly forbidden. Using the internet or the literature is only permitted in a strictly passive manner: one may browse online or offline resources, but may not actively ask questions. Should the committee find out about a violation of these rules, the competitor in question will be disqualified.

Please do not pose or discuss any of these problems at any online forum before the end of the competition, and provide no assistance to the participants in solving the problems. We also ask anyone who notices that this happens to let us know immediately.

Discussion of the problems and presentation of the solutions (in Hungarian) will be held at 4:00 PM CET, 5 November 2024, at the Eötvös Loránd University campus in Lágymányos (1117 Budapest, Pázmány Péter sétány 1/c) in room 3-517. The time and place of the awards ceremony will be announced on the competition's homepage:

`https://www.bolyai.hu/versenyek-schweitzer-miklos-emlekverseny`

Preliminary official solutions (in Hungarian) will also be made available here after the discussion of the problems. Following the awards ceremony, every competitor will be informed via email which of the solutions submitted by them have been deemed complete, essentially correct, a partial solution, or incorrect.

We wish you good luck,

the Schweitzer Committee