# Problems posed in the 2022 Miklós Schweitzer Memorial Competition in Mathematics

## October 28, 2022 – November 7, 2022

**1.** A set $A \subseteq \mathbb{Z}$ is called *irregular* if for any two distinct elements $x, y \in A$, the set $A$ contains no other element of the form $x + k(y - x)$ where $k \in \mathbb{Z}$. Does there exist an infinite irregular set?

**2.** Let $n$ be a positive integer. Suppose that the sum of the matrices $A_1, \ldots, A_n \in \mathbb{R}^{n\times n}$ is the identity matrix but $\sum_{i=1}^n \alpha_i A_i$ is singular whenever at least one of the coefficients $\alpha_i \in \mathbb{R}$ is zero.

a) Show that $\sum_{i=1}^n \alpha_i A_i$ is non-singular provided that $\alpha_i \neq 0$ for all $i$.

b) Prove that if the matrices $A_i$ are symmetric, then each has rank 1.

**3.** Let $f\colon [0,\infty) \to [0,\infty)$ satisfy the following: it is linear between adjacent integers, and for $n = 0, 1, \ldots$

$$f(n) = \begin{cases} 0 & \text{if } 2 \mid n, \\ 4^\ell + 1 & \text{if } 2 \nmid n,\ 4^{\ell-1} \leq n < 4^\ell\ (\ell = 1, 2, \ldots). \end{cases}$$

Let $f^1(x) = f(x)$, and $f^k(x) = f\big(f^{k-1}(x)\big)$ for all integers $k \geq 2$. Determine $\liminf_{k\to\infty} f^k(x)$ and $\limsup_{k\to\infty} f^k(x)$ for Lebesgue almost every $x \in [0,\infty)$.

**4.** For each polynomial $f$ of degree $n$ with integer coefficients, consider the integral

$$\int_{-1}^{1} x^n f(x)\, dx.$$

Let $\alpha_n$ denote the smallest positive real number that can be obtained as the value of such an integral. Determine

$$\lim_{n\to\infty} \frac{\log \alpha_n}{n}.$$

**5.** Is it possible to select a non-degenerate segment from every line in the plane in such a way that they are pairwise disjoint?

**6.** Let $\varepsilon$ be a primitive 7th root of unity. Determine all integers of the form $|\alpha|^2$ for some element $\alpha$ of the cyclotomic field $\mathbb{Q}(\varepsilon)$.

**7.** There are chess pieces located on each vertex of a regular $k$-gon. We are allowed to pick up a piece and move it to the point obtained by reflecting its current position to that of another piece. By repeatedly performing this type of move, for what integers $k \geq 3$ are we able to achieve a situation where all pieces once again occupy the vertices of a regular $k$-gon, but the polygon is of different size than the original?

**8.** Prove that for a suitable choice of the signs $\varepsilon_n = \pm 1$ the function $f(s) = \displaystyle\sum_{n=1}^{\infty} \frac{\varepsilon_n}{n^s} \colon \{\operatorname{Re} s > 1\} \to \mathbb{C}$ accumulates to every complex number at every point $\xi \in \{\operatorname{Re} s = 1\}$ (i.e. for every $z \in \mathbb{C}$ and $\xi \in \{\operatorname{Re} s = 1\}$ there exists a sequence $s_n \to \xi$, $\operatorname{Re} s_n > 1$ such that $f(s_n) \to z$).

**9.** Consider the group formed by the vectors in the plane under addition. Suppose a set $S \subseteq \mathbb{R}^2$ contains a Borel subset of a circular arc that has positive linear measure. Show that $S$ is a generating set of the above group.

**10.** Does there exist a continuous function $f\colon \mathbb{R}\setminus\mathbb{Q} \to \mathbb{R}\setminus\mathbb{Q}$ such that the preimage of each irrational number is of positive Hausdorff dimension?

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The Schweitzer Miklós Competition is open for all students who currently study in Hungary or are Hungarian citizens studying abroad, who do not hold an MSc (or equivalent) degree in mathematics, computer science or related fields obtained in 2021 or earlier. (Those who graduated in 2022 are eligible to participate.)

This year, competitors will have 10 days to solve the problems. Solutions may be submitted by **12:00 PM (noon) CET, November 7, 2022**, via email to the following address: schweitzer.miklos@gmail.com. Please include your full name, affiliation in Hungary, home address and email address when submitting. Solutions should be in **PDF format** (in case of scanned documents, please ensure the legibility of the image), preferably sent in a single letter but **in separate files** for each problem, and written in Hungarian. In exceptional cases, following prior approval, we can accept solutions written in English as well. Any papers submitted after the deadline or that otherwise do not satisfy the above criteria will not be considered by the committee.

Participants may submit solutions to any subset of the problems. Prizes will be handed out to the best performing students by the János Bolyai Mathematical Society. Solving at least three problems correctly will automatically result in receiving a prize or certificate.

Problems are to be solved individually, cooperation of any form is strictly forbidden. Using the internet or the literature is only permitted in a strictly passive manner: one may browse online or offline resources, but may not actively ask questions. Should the committee find out about a violation of these rules, the competitor in question will be disqualified.

Please do not pose or discuss any of these problems at any online forum before the end of the competition, and provide no assistance to the participants in solving the problems. We also ask anyone who notices that this happens to let us know immediately.

Discussion of the problems and presentation of the solutions (in Hungarian) will be held at 4:00 PM CET, November 8, 2022, at the Eötvös Loránd University campus in Lágymányos (1117 Budapest, Pázmány Péter sétány 1/c) in room 4-713. The time and place of the awards ceremony will be announced on the competition's homepage:

http://www.bolyai.hu/schweitzer.htm

Preliminary official solutions (in Hungarian) will also be made available here after the discussion of the problems. Following the awards ceremony, every competitor will be informed via email which of the solutions submitted by them have been deemed complete, essentially correct, a partial solution, or incorrect.

We wish you good luck,

the Schweitzer Committee