Problem I–1

Determine all $k \in \mathbb{N}_0$ for which there exists a function $f\colon \mathbb{N}_0 \to \mathbb{N}_0$ such that $f(2024) = k$ and

$$f(f(n)) \leq f(n+1) - f(n)$$

for all $n \in \mathbb{N}_0$.

Remark. Here $\mathbb{N}_0$ denotes the set of nonnegative integers.

Problem I–2

There is a sheet of paper (like this one) on an infinite blackboard. Marvin secretly chooses a convex 2024-gon P that lies fully on the piece of paper. Tigerin wants to find the vertices of P. In each step, Tigerin can draw a line g on the blackboard that is fully outside the piece of paper, then Marvin replies with the line h parallel to g that is the closest to g which passes through at least one vertex of P. Prove that there exists a positive integer n such that Tigerin can always determine the vertices of P in at most n steps.

Problem I–3

Let ABC be an acute scalene triangle. Choose a circle ω passing through B and C which intersects segments AB and AC again in points $D \neq A$ and $E \neq A$, respectively. Let F be the intersection of BE and CD. Let G be the point on the circumcircle of ABF such that GB is tangent to ω. Similarly, let H be the point on the circumcircle of ACF such that HC is tangent to ω. Prove that there exists a point $T \neq A$, independent of the choice of ω, such that the circumcircle of AGH passes through T.

Problem I–4

For any positive integer n, let $\sigma(n)$ denote the sum of positive divisors of n. Determine all polynomials P with integer coefficients such that $P(k)$ is divisible by $\sigma(k)$ for all positive integers k.

Time: 5 hours
Time for questions: 60 min
Each problem is worth 8 points.
The order of the problems does not depend on their difficulty.