Többváltozós diszkrét momentum problémák és alkalmazásaik

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The univariate discrete moment problem (DMP)

Let X be a random variable with the support $I = \{z_0, z_1, \ldots, z_n\}$.

Let $p_j = P(X = z_j)$, $f(z_j) = f_j$, $j = 0, 1, \ldots, n$, where the probabilities p_js are unknown. But known are the power moments: $\mu_i = E[X^i] = z_0^i p_0 + z_1^i p_1 + \cdots + z_n^i p_n$, $i = 0, 1, \ldots, m$

The LP corresponding to the DMP is

$$
\begin{array}{llllll}
\min(\max) & E[f(X)] = f_0 p_0 & + f_1 p_1 & + \cdots & + f_n p_n & \\
\text{subject to} & & & & & \\
& p_0 & + p_1 & + \cdots & + p_n & = \mu_0 (= 1) \\
& z_0 p_0 & + z_1 p_1 & + \cdots & + z_n p_n & = \mu_1 \\
& z_0^2 p_0 & + z_1^2 p_1 & + \cdots & + z_n^2 p_n & = \mu_2 \\
& & & & \vdots & \\
& z_0^m p_0 & + z_1^m p_1 & + \cdots & + z_n^m p_n & = \mu_m \\
& p_0, & p_1, & \ldots & p_n & \geq 0.
\end{array}
$$

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z_0^m p_0 & +z_1^m p_1 & +\cdots & +z_n^m p_n & = & \mu_m \\
p_0, & p_1, & \ldots & p_n & \geq & 0.
\end{array} \qquad (1)$$

The multivariate power DMP

Let $\boldsymbol{X} = (X_1, \ldots, X_s)$ be a random vector with support $Z = Z_1 \times \cdots \times Z_s$, where $Z_j = \{z_{j0}, \ldots, z_{jn_j}\}$.

The $(\alpha_1, \ldots, \alpha_s)$-order power moment of the random vector is defined as

$$\mu_{\alpha_1 \ldots \alpha_s} = E\left[X_1^{\alpha_1} \cdots X_s^{\alpha_s}\right],$$

Let us denote the unknown probabilities by

$$p_{i_1 \ldots i_s} = P(X_1 = z_{1 i_1}, \ldots, X_s = z_{s i_s}),\ 0 \le i_j \le n_j,\ j = 1, \ldots, s. \tag{2}$$

Let $f(\boldsymbol{z})$, $\boldsymbol{z} \in Z$ be a function and let

$$f_{i_1 \ldots i_s} = f(z_{1 i_1}, \ldots, z_{s i_s}).$$

The multivariate power DMP

$$E[f(\boldsymbol{X})] = \sum_{i_1=0}^{n_1} \cdots \sum_{i_s=0}^{n_s} f_{i_1 \dots i_s} p_{i_1 \dots i_s}$$

The power MDMP can be formulated by the following LP:

$$\begin{aligned} & \min(\max) \sum_{i_1=0}^{n_1} \cdots \sum_{i_s=0}^{n_s} f_{i_1 \dots i_s} p_{i_1 \dots i_s} \\ & \text{subject to} \\ & \sum_{i_1=0}^{n_1} \cdots \sum_{i_s=0}^{n_s} z_{1 i_1}^{\alpha_1} \cdots z_{s i_s}^{\alpha_s} p_{i_1 \dots i_s} = \mu_{\alpha_1 \dots \alpha_s} \\ & \text{for } (\alpha_1, \dots, \alpha_s) \in H, \\ & p_{i_1 \dots i_s} \geq 0, \ \text{all } i_1, \dots, i_s. \end{aligned} \tag{3}$$

DMP vs. MDMP

- The coefficient matrices of the DMP as well as the MDMP are ill-conditioned
- The dual feasible bases of the power DMP for special sets of objective functions are known: Prékopa (1990b).
- Then the numerically stable dual method of Prékopa (1990b) can be applied.
- Unfortunately, this method cannot be generalized for the multivariate case: only a smaller set of dual feasible bases are known. See Prékopa (1998), Mádi-Nagy and Prékopa (2004) and Mádi-Nagy (2005, 2009).
- The dual feasible bases provide us with bounds for the MDMP, however, the optimum of the problem usually cannot be found.

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Assumptions for the objective function

Suppose that the function $f(\boldsymbol{z}),\ \boldsymbol{z} \in Z$ has nonnegative divided differences of total order $m+1$, and, in addition, in each variable z_j it has nonnegative divided differences of order m_j.

If $f(\boldsymbol{z}),\ \boldsymbol{z} \in Z$ is derived from a function $\overline{f}(\boldsymbol{z})$ defined in $\overline{Z} = [z_{10}, z_{1n_1}] \times \cdots \times [z_{s0}, z_{sn_s}]$ by taking $f(\boldsymbol{z}) = \overline{f}(\boldsymbol{z}),\ \boldsymbol{z} \in Z$ and $\overline{f}(\boldsymbol{z})$ has continuous, nonnegative derivatives of order $(k_1, \ldots, k_s)$ in the interior of $\overline{Z}$, then all divided differences of $f(\boldsymbol{z}),\ \boldsymbol{z} \in Z$ of order $(k_1, \ldots, k_s)$ are nonnegative.

Natural generalization

$$
\begin{aligned}
H \quad = \quad & \{(\alpha_1,\ldots,\alpha_s)|\ 0 \le \alpha_j,\ \alpha_j \text{ integer},\ \alpha_1+\cdots+\alpha_s \le m; \\
& \text{or } \alpha_j = 0,\ j=1,\ldots,k-1,k+1,\ldots,s, \\
& m \le \alpha_k \le m_k,\ \ k=1,\ldots,s\}
\end{aligned} \tag{4}
$$

9	•	○	○	○	○	○	○	○	○	○
8	•	○	○	○	○	○	○	○	○	○
7	○	○	○	○	○	○	○	○	○	○
6	○	○	○	○	○	○	○	○	○	○
5	○	○	○	○	○	○	○	○	○	○
4	•	○	○	○	○	○	○	○	○	○
3	*	*	○	○	○	○	○	○	○	○
2	*	*	*	○	○	○	○	○	○	○
1	*	*	*	*	○	○	○	○	○	○
0	*	*	*	*	•	○	○	○	•	•
	0	1	2	3	4	5	6	7	8	9

(a)

9	•	•	○	○	○	•	*	*	*	*
8	○	○	○	○	○	○	*	*	*	*
7	○	○	○	○	○	○	○	*	*	*
6	○	○	○	○	○	○	○	○	*	*
5	○	○	○	○	○	○	○	○	○	•
4	○	○	○	○	○	○	○	○	○	○
3	○	○	○	○	○	○	○	○	○	○
2	○	○	○	○	○	○	○	○	○	○
1	○	○	○	○	○	○	○	○	○	•
0	○	○	○	○	○	○	○	○	○	•
	0	1	2	3	4	5	6	7	8	9

(b)

Dual feasible bases for the min (a) and max (b) problems.
$Z_1 = Z_2 = \{0,\ldots,9\}.\ \ m=4,\ \ m_1 = m_2 = 6$

More bases for $s = 2$

9	∗	∗	∘	∘	∘	∘	∘	∘	∗	∗
8	∗	∘	∘	∘	∘	∘	∘	∘	∘	∗
7	∘	∘	∘	∘	∘	∘	∘	∘	∘	∘
6	•	∘	∘	∘	∘	∘	∘	∘	∘	∘
5	•	∘	∘	∘	∘	∘	∘	∘	∘	∘
4	∘	∘	∘	∘	∘	∘	∘	∘	∘	∘
3	∘	∘	∘	∘	∘	∘	∘	∘	∘	∘
2	•	∘	∘	∘	∘	∘	∘	∘	∘	∘
1	∗	∗	∘	∘	∘	∘	∘	∘	∘	∗
0	∗	∗	•	∘	∘	•	•	∘	∗	∗
	0	1	2	3	4	5	6	7	8	9

(a)

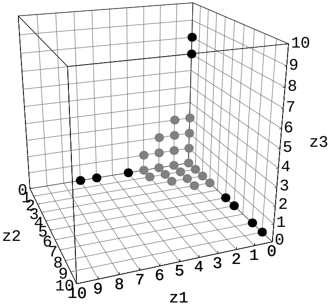
9	∗	∗	∘	∘	∘	∘	∘	∘	∘	∗
8	∘	∘	∘	∘	∘	∘	∘	∘	∘	∘
7	•	∘	∘	∘	∘	∘	∘	∘	∘	∘
6	•	∘	∘	∘	∘	∘	∘	∘	∘	∘
5	∘	∘	∘	∘	∘	∘	∘	∘	∘	∘
4	∘	∘	∘	∘	∘	∘	∘	∘	∘	∘
3	•	∘	∘	∘	∘	∘	∘	∘	∘	∘
2	∗	∗	∘	∘	∘	∘	∘	∘	∘	∘
1	∗	∗	∗	∘	∘	∘	∘	∘	∘	∗
0	∗	∗	∗	•	∘	∘	•	•	∘	∗
	0	1	2	3	4	5	6	7	8	9

(b)

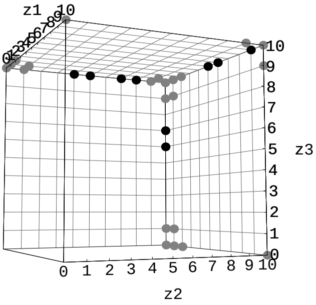
Dual feasible bases for the min (a) and max (b) problems.
$Z_1 = Z_2 = \{0, \ldots, 9\}$. $m = 4$, $m_1 = m_2 = 6$

Other way of the higher dimensional generalization

$$H = \{(\alpha_1, 0, \ldots, 0, \alpha_j, 0, \ldots, 0) |\ 0 \le \alpha_1, \alpha_j,\ \alpha_1, \alpha_j \text{ integer}, \alpha_1 + \alpha_j \le m, j = 2, \ldots, s\} \cup \{(0, \ldots, 0, \alpha_j, 0, \ldots, 0) |\ m+1 \le \alpha_j \le m_j,\ \alpha_j \text{ integer}, j = 1, \ldots, s\}. \quad (5)$$

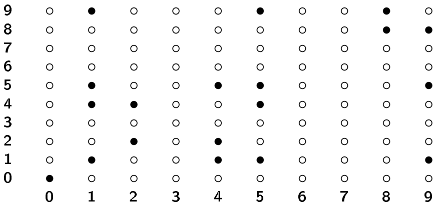


(a)

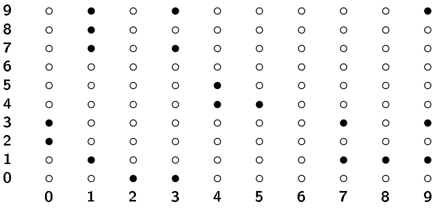


(b)

However



(a)



(b)

Optimal bases in Example 6.2 in Mádi-Nagy and Prékopa (2004) for the min (a) and max (b) problems. $Z_1 = Z_2 = \{0, \dots, 9\}$. $m = 4, \; m_1 = m_2 = 6$

Polynomial basis: monomials

Let us consider the following vector:

$$\boldsymbol{b}(\boldsymbol{z}) = \boldsymbol{b}(z_1, \ldots, z_s) = \begin{pmatrix} 1 \\ z_1 \\ z_1^2 \\ \vdots \\ z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_s^{\alpha_s} \\ \vdots \\ z_s^m \end{pmatrix}, \text{ where } \alpha_1 + \cdots + \alpha_s \leq m. \tag{6}$$

The components of $\boldsymbol{b}(\boldsymbol{z})$ of (6) are the monomial basis of the s-variate polynomials of degree at most m.

The embedding polynomial basis

Let us use the following notation for the compact matrix form of (3):

$$\begin{aligned} & \min(\max) \quad \boldsymbol{f}^T\boldsymbol{p} \\ \text{subject to} & \\ & A\boldsymbol{p} = \boldsymbol{b} \\ & \boldsymbol{p} \geq 0. \end{aligned} \tag{7}$$

Regarding the columns of the coefficient matrix A in (7), they can be formulated as

$$\boldsymbol{a}_{i_1\dots i_s} = \boldsymbol{b}(z_{1i_1}, \dots, z_{si_s}).$$

The right-hand side vector can also be written as

$$\boldsymbol{b} = E\left(\boldsymbol{b}(X_1, \dots, X_s)\right).$$

Other polynomial bases

Let us consider another basis of the s-variate polynomials of degree at most m:

$$p_{0\dots0}(\boldsymbol{z}),\ p_{1\dots0}(\boldsymbol{z}),\dots,\ p_{\alpha_1\dots\alpha_s}(\boldsymbol{z}),\dots,\ p_{0\dots m}(\boldsymbol{z}) \tag{8}$$

Let

$$\bar{\boldsymbol{b}}(\boldsymbol{z}) = \bar{\boldsymbol{b}}(z_1,\dots,z_s) = \begin{pmatrix} p_{0\dots0}(\boldsymbol{z}) \\ p_{1\dots0}(\boldsymbol{z}) \\ \vdots \\ p_{\alpha_1\dots\alpha_s}(\boldsymbol{z}) \\ \vdots \\ p_{0\dots m}(\boldsymbol{z}) \end{pmatrix}, \text{ where } \alpha_1,\dots,\alpha_s \le m, \tag{9}$$

Other polynomial bases

$$\bar{\boldsymbol{a}}_{i_1 \dots i_s} := \bar{\boldsymbol{b}}(z_{1 i_1}, \dots, z_{s i_s}),$$
$$\bar{\boldsymbol{b}} := E\left(\bar{\boldsymbol{b}}(X_1, \dots, X_s)\right).$$

The system of linear equations $\bar{A}\boldsymbol{p} = \bar{\boldsymbol{b}}$ is equivalent to the system $A\boldsymbol{p} = \boldsymbol{b}$ of (7), since there exists an invertible matrix T such that

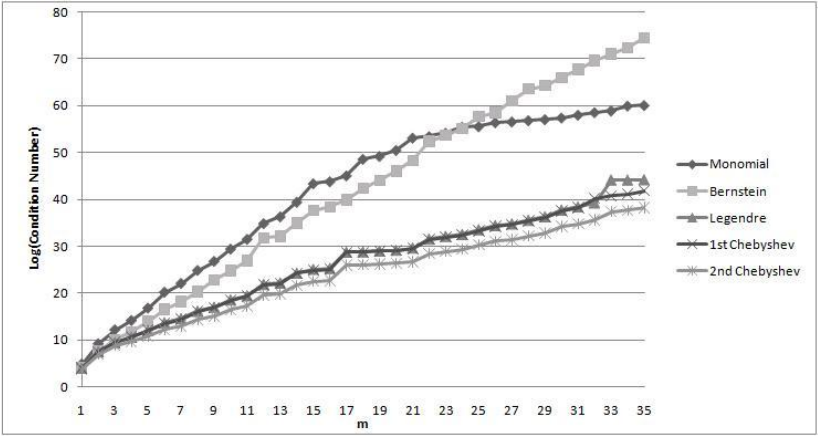
$$\bar{A} = TA \text{ and } \bar{\boldsymbol{b}} = T\boldsymbol{b}.$$

Our aim is to find out which basis (8) yields a significantly better conditioned matrix $\bar{A}$. By the use of this basis we can solve

$$\begin{array}{lrcl} & \min(\max) & \boldsymbol{f}^T\boldsymbol{p} & \\ \text{subject to} & & & \\ & \bar{A}\boldsymbol{p} & = & \bar{\boldsymbol{b}} \\ & \boldsymbol{p} & \geq & \boldsymbol{0}, \end{array} \tag{10}$$

instead of problem (7), in a numerically more stable way.

Condition numbers of randomly generated bases



Solution algorithm

Solution Algorithm

Step 1. Execution of the basis transformation from problem (7) to problem (10) by the use of high precision arithmetic.

Step 2. Solution of problem (10) by a regular LP solver using dual simplex method.

Step 3. Getting the subscripts of the columns of the optimal basis. Checking the primal and dual feasibility by the use of high precision arithmetic with problem (7). Calculating the objective function value.

Multivariate utility function

Let $k \geq 1$ and $D = \{(z_1, \ldots, z_s) | e^{g_j(z_j)} > 2,\ j = 1, \ldots, s\}$. We define the utility function u as:

$$u(z_1, \ldots, z_s) := log \left[k(e^{g_1(z_1)} - 1) \cdots (e^{g_s(z_s)} - 1) - 1 \right], \qquad (11)$$

where for every $(z_1, \ldots, z_n) \in D$ the following conditions hold:

$$\begin{aligned} g_j'(z_j) &> 0 \\ g_j^{(i)}(z_j) &\geq 0, \text{ if } i > 1 \text{ and is odd} \\ g_j^{(i)}(z_j) &\leq 0, \text{ if } i \text{ is even} \\ & j = 1, \ldots, s. \end{aligned} \qquad (12)$$

The $g_j(z_j)$ functions can be chosen, e.g.,

$$\begin{aligned} a \log\left(1 + \frac{z}{b}\right), &\quad \text{where } a > 0,\ b > 0, \\ ae^{-bz}, &\quad \text{where } a > 0,\ b > 0, \\ a_n z^n + \cdots + a_1 z + a_0, &\quad \text{with suitably chosen coefficients.} \end{aligned}$$

Arrow-Pratt type properties of $u(z_1, \dots, z_s)$

- It is obvious that functions (11) are strictly increasing in each variable.
- The functions (11) are concave on D.
- For every $\boldsymbol{z} = (z_1, \dots, z_s) \in D$ we have

$$\frac{\partial^{i_1+\cdots+i_s} u(z_1, \dots, z_s)}{\partial z_1^{i_1} \cdots \partial z_s^{i_s}} > 0, \text{ if } i_1 + \cdots + i_s \text{ is odd,}$$

and

$$\frac{\partial^{i_1+\cdots+i_s} u(z_1, \dots, z_s)}{\partial z_1^{i_1} \cdots \partial z_s^{i_s}} < 0, \text{ if } i_1 + \cdots + i_s \text{ is even.} \tag{13}$$

Hence, $E[u(X_1, \dots, X_n)]$ can be bounded based on the MDMP results.

Bounding moment generating function

The joint moment generating function of the random variables $X_1, \ldots, X_s$ is defined as

$$M(t_1, \ldots, t_s) = E[e^{t_1 X_1 + \cdots + t_s X_s}].$$

If it is finite in an open neighborhood around the origin, then M completely determines the distribution of $\boldsymbol{X} = (X_1, \ldots, X_s)$. Other interesting properties are:

$M(0, \ldots, 0, t_i, 0, \ldots, 0) = M_i(t_i), \ i = 1, \ldots, s$ and

$$\frac{\partial^{\alpha_1 + \cdots \alpha_s} M}{\partial t_1{}^{\alpha_1} \cdots \partial t_s{}^{\alpha_s}}(0, \ldots, 0) = \mu_{\alpha_1 \ldots \alpha_s}.$$

If we assume that $\boldsymbol{X}$ has a finite support, then we can use the MDMP for bounding the value of the joint moment generating function for certain values of $(t_1, \ldots, t_s)$ in terms of the (mixed) power moments of $\boldsymbol{X}$.

Cross-binomial moments

Assume that we have n arbitrary events. We can subdivide them into s subsequences. Let the jth subsequence be designated as $A_{j1}, \ldots, A_{jn_j}$, $j = 1, \ldots, s$. Certainly, $n_1 + \cdots + n_s = n$. Let the random variable X_j with the support $Z_j = \{0, 1, \ldots, n_j\}$ be the number of events which occur in the jth sequence, $j = 1, \ldots, s$. In case of event sequences

$$S_{\alpha_1 \ldots \alpha_s} = \sum_{\substack{1 \le i_{j1} < \cdots < i_{j\alpha_j} \le n_j, \\ j=1,\ldots,s}} P[A_{1i_{11}} \cap \cdots \cap A_{1i_{1\alpha_1}} \cap \cdots \cap A_{si_{s1}} \cap \cdots \cap A_{si_{s\alpha_s}}], \tag{14}$$

It can be proven that

$$S_{\alpha_1 \ldots \alpha_s} = E\left[\binom{X_1}{\alpha_1} \cdots \binom{X_s}{\alpha_s}\right],$$

Binomial MDMP

$$\min(\max) \sum_{i_1=0}^{n_1} \cdots \sum_{i_s=0}^{n_s} f_{i_1 \ldots i_s} p_{i_1 \ldots i_s}$$

subject to

$$\sum_{i_1=0}^{n_1} \cdots \sum_{i_s=0}^{n_s} \binom{i_1}{\alpha_1} \cdots \binom{i_s}{\alpha_s} p_{i_1 \ldots i_s} = S_{\alpha_1 \ldots \alpha_s} \tag{15}$$

$$\text{for } \alpha_1 + \cdots + \alpha_s \leq m$$

$$p_{i_1 \ldots i_s} \geq 0, \text{ all } i_1, \ldots, i_s.$$

Remark: it is an MDMP in special polynomial basis

If we would like to give bounds for the probability of the union all the n events, then we have to consider

$$f(z_1, \ldots, z_s) = \begin{cases} 0 \text{ if } (z_1, \ldots, z_s) = (0, \ldots, 0) \\ 1 \text{ otherwise} \end{cases} \tag{16}$$

The method of calculation

- The CDF can be interpreted as

$$F(x_1, \dots, x_n) = P(\xi_1 < x_1, \dots, \xi_n < x_n) = 1 - P(A_1 \cup \dots \cup A_n), \tag{17}$$

  where $A_i = \{\xi_i \geq x_i\},\ i = 1, \dots, n.$
- Assume that the value of the CDF can be calculated easily up to m dimensions (usually $n >> m$). Hence,

$$P(A_{i_1} \cap \dots \cap A_{i_k}), \quad 1 \leq i_1 < \dots < i_k \leq n,\ k \leq m, \tag{18}$$

  can be found.
- Then we can calculate lower and upper bounds on the values of (17) by use of binomial MDMPs (15) with objective function (16).

Examples: multivariate Dirichlet distribution

Example 1 of Bukszár, Mádi-Nagy and Szántai (2012): The 3-rd order bounds for the Dirichlet CDF:

	bound value	CPU in seconds
Multitree lower	0.103087	0.04
Multitree upper	0.293917	1.40
Aggregated lower	0.262142	1.18+0.00
Aggregated upper	0.305193	1.18+0.00
Bivariate lower	0.262142	1.18+0.01
Bivariate upper	0.297989	1.18+0.01
Multivariate lower	0.262142	1.18+0.01
Polynomial lower	0.262142	1.18+8.30
Polynomial upper	0.295224	1.18+9.21

Bivariate Bonferroni-Type upper bound in case of $m = 4$

Let $A_1, \ldots, A_N$ and $B_1, \ldots, B_M$ be two sequences of events.

- $\nu_N(A)$: the number of those A_i that occur,
- $\nu_M(B)$: the number of those B_j that occur.

$$P(\nu_N(A) \geq 1, \nu_M(B) \geq 1) \leq S_{11}$$

$$-\frac{2(NM - N + M - 2)}{NM(M-1)} S_{12} - \frac{2(NM - M + N - 2)}{NM(N-1)} S_{21} \qquad (19)$$

$$+\frac{6}{NM(M-1)} S_{13} + \frac{6}{NM(N-1)} S_{31} + \frac{4}{NM} S_{22}.$$

Network reliability

Bukszár, Mádi-Nagy and Szántai (2012).
Consider a compact, directed and acyclic network $(\mathcal{N}, \mathcal{A})$. Assume that

- $\mathcal{N} = \{c_1, \ldots, c_n\}$ is the set of nodes, and
- $\mathcal{A} \subset \mathcal{N} \times \mathcal{N}$ is the set of arcs,
- c_1 is the single start node and c_n is the single terminal node,
- each arc is alive with probability p independently.

The probability of the event that one can get from the start node c_1 to the terminal node c_n along living arcs is called the reliability of the network.

Network reliability

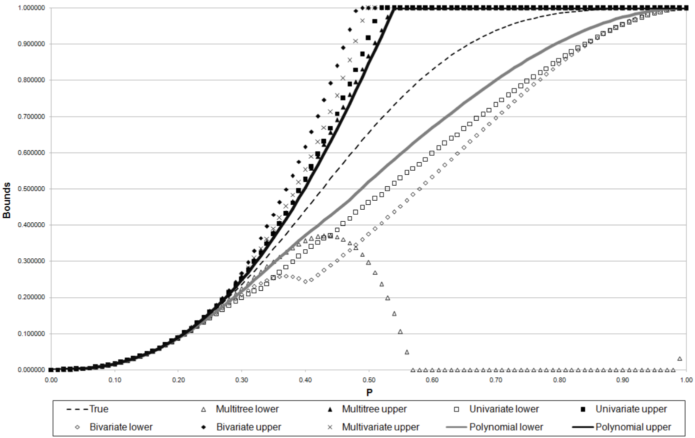
- Consider all paths leading from the start node c_1 to the terminal node c_n.
- Denote these paths by $P_1, \dots, P_N$ and denote the event that all arcs along the path P_i are alive by $A_i, i = 1, \dots, N$.

With these notations the reliability of the network equals

$$P(A_1 \cup \cdots \cup A_N).$$

The probability of the event that all arcs are alive along a few, say not more than five, paths can be calculated quickly.
In the numerical example 8 nodes, 16 arcs, 23 paths. The exact value of the reliability can be calculated for the comparison.

Bounds based on $P(A_i)$, $P(A_i \cap A_j)$, $P(A_i \cap A_j \cap A_k)$



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