A logaritmikus legkisebb négyzetek módszerének karakterizációi

Csató László
laszlo.csato@uni-corvinus.hu

MTA Számítástechnikai és Automatizálási Kutatóintézet (MTA SZTAKI)
Operációkutatás és Döntési Rendszerek Kutatócsoport

Budapesti Corvinus Egyetem (BCE)
Operációkutatás és Aktuáriustudományok Tanszék

XXXII. Magyar Operációkutatási Konferencia

Cegléd
2017. június 15.

Outline

What is the goal?

Inconsistency in pairwise comparisons

- Inconsistency: alternative A is two times better than alternative B, alternative B is three times better than alternative C, but alternative A is NOT six $(= 2 \times 3)$ times better than alternative C
- What is the best approximation of these inconsistent preferences?
- Different answers: Eigenvector Method (*EM*), *Logarithmic Least Squares Method / geometric mean* (*LLSM*), Least Squares Method, Preference Weighted Least Squares, Least Absolute Error etc.

Axiomatic discussion

- Reasonable properties are suggested and examined
- Some characterizations are known, mainly for *LLSM*
- We provide a new axiomatization of *LLSM* as well as of the ranking induced by this method

Pairwise comparison matrices and weighting methods

Pairwise comparison matrix

Matrix $\mathbf{A} = [a_{ij}] \in \mathbb{R}_+^{n \times n}$ is a *pairwise comparison matrix* if $a_{ji} = 1/a_{ij}$ for all $1 \leq i, j \leq n$.
The set of all pairwise comparison matrices of size $n \times n$ is denoted by $\mathcal{A}^{n \times n}$.

Consistency

Pairwise comparison matrix $\mathbf{A} = [a_{ij}]$ is *consistent* if $a_{ik} = a_{ij} a_{jk}$ for all $1 \leq i, j, k \leq n$.

Weight vector

Vector $\mathbf{w} = [w_i] \in \mathbb{R}_+^n$ is a *weight vector* if $\sum_{i=1}^n w_i = 1$.
The set of all weight vectors of size n is denoted by $\mathcal{R}^n$.

Weighting method

Function $f : \mathcal{A}^{n \times n} \to \mathcal{R}^n$ is a *weighting method*.

Two weighting methods

Eigenvector Method (*EM*)

The *Eigenvector Method* is the function $\mathbf{A} \to \mathbf{w}^{EM}(\mathbf{A})$ such that

$$\mathbf{A}\mathbf{w}^{EM}(\mathbf{A}) = \lambda_{\max}\mathbf{w}^{EM}(\mathbf{A}),$$

where $\lambda_{\max}$ denotes the maximal eigenvalue of matrix $\mathbf{A}$.

Logarithmic Least Squares Method (*LLSM*)

The *Logarithmic Least Squares Method* is the function $\mathbf{A} \to \mathbf{w}^{LLSM}(\mathbf{A})$ such that the weight vector $\mathbf{w}^{LLSM}(\mathbf{A})$ is the optimal solution of the problem:

$$\min_{\mathbf{w} \in \mathbb{R}^n_+, \sum_{i=1}^n w_i = 1} \sum_{i=1}^n \sum_{j=1}^n \left[\log a_{ij} - \log \left(\frac{w_i}{w_j} \right) \right]^2 . \quad (1)$$

LLSM is sometimes called *(row) geometric mean* since the solution of (1) can be computed as $w_i = \prod_{j=1}^n a_{ij}^{1/n} / \left(\sum_{i=1}^n \prod_{j=1}^n a_{ij}^{1/n} \right)$.

Axioms for a weighting method

Correctness (CR)

Let $\mathbf{A} \in \mathcal{A}^{n \times n}$ be a consistent pairwise comparison matrix. Weighting method $f: \mathcal{A}^{n \times n} \to \mathcal{R}^n$ is *correct* if $f_i(\mathbf{A})/f_j(\mathbf{A}) = a_{ij}$ for all $1 \leq i,j \leq n$.

Transformation of consistency reconstruction

Let $\mathbf{A} \in \mathcal{A}^{n \times n}$ be a pairwise comparison matrix and $1 \leq i,j,k \leq n$ be three different alternatives. A *transformation of consistency reconstruction* on (i,j,k) by α provides the pairwise comparison matrix $\hat{\mathbf{A}} \in \mathcal{A}^{n \times n}$ such that $\hat{a}_{ij} = \alpha a_{ij}$, $\hat{a}_{jk} = \alpha a_{jk}$, $\hat{a}_{ik} = a_{ik}/\alpha$ and $\hat{a}_{\ell m} = a_{\ell m}$ for all other elements.

Independence of consistency reconstruction (ICR)

Let $\mathbf{A}, \hat{\mathbf{A}} \in \mathcal{A}^{n \times n}$ be two pairwise comparison matrices such that $\hat{\mathbf{A}}$ is obtained from $\mathbf{A}$ through a transformation of consistency reconstruction. Weighting method $f: \mathcal{A}^{n \times n} \to \mathcal{R}^n$ is *independent of consistency reconstruction* if $f(\mathbf{A}) = f\left(\hat{\mathbf{A}}\right)$.

Analysis of weighting methods

Lemma

The Eigenvector Method and the Logarithmic Least Squares Method satisfy correctness.

Lemma

The Eigenvector Method violates independence of consistency reconstruction.

Proof

Consider the following pairwise comparison matrices:

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 8 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1/8 & 1 & 1 & 1 \end{bmatrix} \text{ and } \hat{\mathbf{A}} = \begin{bmatrix} 1 & 2 & 1 & 4 \\ 1/2 & 1 & 1 & 2 \\ 1 & 1 & 1 & 1 \\ 1/4 & 1/2 & 1 & 1 \end{bmatrix}.$$

$\hat{\mathbf{A}}$ can be obtained from $\mathbf{A}$ through a transformation of consistency reconstruction on $(1, 2, 4)$ by $\alpha = 2$, but $\mathbf{w}^{EM}(\mathbf{A}) \neq \mathbf{w}^{EM}(\hat{\mathbf{A}})$.

Characterization of *LLSM* as a weighting method

Lemma

The Logarithmic Least Squares Method satisfies independence of consistency reconstruction.

Theorem (first main result)

The Logarithmic Least Squares Method is the unique weighting method satisfying correctness and independence of consistency reconstruction.

Proof

1. Check that *LLSM* satisfies both axioms (see the previous lemmata).
2. Uniqueness: Let $P_i = \sqrt[n]{\prod_{k=1}^{n} a_{i,k}}$. Define the pairwise comparison matrix $\mathbf{A}^{(n-1,n)} \in \mathcal{A}^{n \times n}$ such that $a_{1,n-1}^{(n-1,n)} = \alpha_{n-1,n} a_{1,n-1}$, $a_{1,n}^{(n-1,n)} = a_{1,n} / \alpha_{n-1,n}$, $a_{n-1,n}^{(n-1,n)} = \alpha_{n-1,n} a_{n-1,n}$ and $a_{i,j} = a_{i,j}^{(n-1,n)}$ for all other elements, where $\alpha_{n-1,n} = P_{n-1} / \left(P_n a_{n-1,n} \right)$. So $a_{n-1,n}^{(n-1,n)} = P_{n-1} / P_n$. Follow this procedure to get the consistent matrix $\mathbf{A}^{(2,3)} \in \mathcal{A}^{n \times n}$.

Independence of the axioms

Lemma

CR and ICR are logically independent axioms.

Proof

It is shown that there exist scoring methods, which satisfy one axiom, but do not meet the other:

1. *CR*: Eigenvector Method;
2. *ICR*: flat method, that is, $f_i(\mathbf{A}) = f_j(\mathbf{A}) = 1/n$ for all $1 \leq i, j \leq n$.

Ranking of the alternatives

- Weighting methods are often used only to derive a *ranking* of the alternatives
- Ranking $\succeq$ is a complete ($i \succeq j$ or $i \preceq j$ for all $1 \leq i, j \leq n$) and transitive (for all $1 \leq i, j, k \leq n$: if $i \succeq j$ and $j \succeq k$, then $i \succeq k$) weak order on the set of alternatives
- The set of all rankings on n alternatives is denoted by $\mathfrak{R}^n$

Ranking method

Function $g : \mathcal{A}^{n \times n} \to \mathfrak{R}^n$ is a *ranking method*.

Example

Logarithmic Least Squares Ranking Method is the function $\mathbf{A} \to \succeq_{\mathbf{A}}^{LLSM}$ such that $i \succeq_{\mathbf{A}}^{LLSM} j$ if $w_i^{LLSM}(\mathbf{A}) \geq w_i^{LLSM}(\mathbf{A})$ for all $\mathbf{A} \in \mathcal{A}^{n \times n}$.

Axioms for a ranking method I.

Anonymity (*ANO*)

Let $\mathbf{A} = [a_{ij}] \in \mathcal{A}^{n \times n}$ be a pairwise comparison matrix, $\sigma : N \to N$ be a permutation on the set of alternatives N, and $\sigma(\mathbf{A}) = [\sigma(a)_{ij}] \in \mathcal{A}^{n \times n}$ be the pairwise comparison matrix obtained from $\mathbf{A}$ by this permutation such that $\sigma(a)_{ij} = a_{\sigma(i)\sigma(j)}$. Ranking method $g : \mathcal{A}^{n \times n} \to \mathfrak{R}^n$ is *anonymous* if $i \succeq_{\mathbf{A}}^{g} j \iff \sigma(i) \succeq_{\sigma(\mathbf{A})}^{g} \sigma(j)$ for all $1 \leq i, j \leq n$.

Responsiveness (*RES*)

Let $\mathbf{A}, \mathbf{A}' \in \mathcal{A}^{n \times n}$ be two pairwise comparison matrices and $1 \leq i, j \leq n$ be two different alternatives such that $\mathbf{A}$ and $\mathbf{A}'$ are identical but $a'_{ij} > a_{ij}$. Ranking method $g : \mathcal{A}^{n \times n} \to \mathfrak{R}^n$ is called *responsive* if $i \succeq_{\mathbf{A}}^{g} j \Rightarrow i \succ_{\mathbf{A}'}^{g} j$.

Axioms for a ranking method II.

Aggregation of pairwise comparison matrices

Let $\mathbf{A}^{(1)} = \left[a_{ij}^{(1)} \right] \in \mathcal{A}^{n \times n}$, $\mathbf{A}^{(2)} = \left[a_{ij}^{(2)} \right] \in \mathcal{A}^{n \times n}$, $\dots$, $\mathbf{A}^{(k)} = \left[a_{ij}^{(k)} \right] \in \mathcal{A}^{n \times n}$ be some pairwise comparison matrices. Their *aggregate* is the pairwise comparison matrix $\mathbf{A}^{(1)} \oplus \mathbf{A}^{(2)} \oplus \dots \oplus \mathbf{A}^{(k)} = \left[\sqrt[k]{a_{ij}^{(1)} a_{ij}^{(2)} \cdots a_{ij}^{(k)}} \right] \in \mathcal{A}^{n \times n}$.

Remark

Geometric mean is the only quasiarithmetic mean which meets reciprocity and positive homogeneity.

Aggregation invariance (*AI*)

Let $\mathbf{A}^{(1)}, \mathbf{A}^{(2)}, \dots, \mathbf{A}^{(k)} \in \mathcal{A}^{n \times n}$ be some pairwise comparison matrices. Let $g : \mathcal{A}^{n \times n} \to \mathfrak{R}^n$ be a ranking method such that $i \succeq_{\mathbf{A}^{(\ell)}}^g j$ for all $1 \leq \ell \leq k$. g is called *aggregation invariant* if $i \succeq_{\mathbf{A}^{(1)} \oplus \mathbf{A}^{(2)} \oplus \dots \oplus \mathbf{A}^{(k)}}^g j$, furthermore, $i \succ_{\mathbf{A}^{(1)} \oplus \mathbf{A}^{(2)} \oplus \dots \oplus \mathbf{A}^{(k)}}^g j$ if at least one of the individual preferences is strict.

Characterization of *LLSM* as a ranking method

Theorem (second main result)

The Logarithmic Least Squares Ranking Method is the unique ranking method satisfying anonymity, responsiveness and aggregation invariance.

Lemma

ANO, RES and AI are logically independent axioms.

Proof

It is shown that there exist ranking methods, which satisfy exactly two properties from the set *ANO*, *RES* and *AI*, but differ from the Logarithmic Least Squares Ranking Method (therefore violate the third axiom):

1. *RES* and *AI*: ranking by indices, $i \succ^g_{\mathbf{A}} j$ for all $\mathbf{A} \in \mathcal{A}^{n \times n}$ if $i < j$
2. *ANO* and *AI*: flat ranking, $i \sim^g_{\mathbf{A}} j$ for all $i, j \in N$ and $\mathbf{A} \in \mathcal{A}^{n \times n}$
3. *ANO* and *RES*: ranking by arithmetic means, $i \succeq^g_{\mathbf{A}} j$ for all $i, j \in N$ and $\mathbf{A} \in \mathcal{A}^{n \times n}$ if $\sum_{k=1}^{n} a_{ik} \geq \sum_{k=1}^{n} a_{jk}$

Conclusions

Observations

- Reversed Logarithmic Least Squares Ranking Method can be characterized by anonymity, aggregation invariance and *negative* responsiveness ($i \preceq^g_{\mathbf{A}} j \Rightarrow i \prec^g_{\mathbf{A}'} j$ if $a'_{ij} > a_{ij}$))
- Conjecture: other quasiarithmetic means can be characterized by *ANO*, *RES* and an appropriate version of *AI*
- Axiomatizations essentially depend on *IIC* and *AI*, respectively

Topics for future research

- Eigenvector Ranking Method and responsiveness
- Other characterizations of weighting and ranking methods
- Extension to the incomplete case

It is NOT suggested that these axioms should be accepted!

A choice of axioms is not purely a subjective task. It is usually expected to achieve some definite aim – some specific theorem or theorems are to be derivable from the axioms – and to this extent the problem is exact and objective. But beyond this there are always other important desiderata of a less exact nature: the axioms should not be too numerous, their system is to be as simple and transparent as possible, and each axiom should have an immediate intuitive meaning by which its appropriateness can be judged directly.

(Neumann, J. és Morgenstern, O.: *Theory of Games and Economic Behavior*)

Köszönöm szépen a figyelmet!