Problems of the 2019 Miklós Schweitzer Memorial Competition in Mathematics

25th October 2019 – 4th November 2019

Problem 1. Prove that if every subspace of a Hausdorff space X is σ-compact then X is countable.

Problem 2. Let R be a noncommutative finite ring with multiplicative identity element 1. Show that if the subring generated by $I \cup \{1\}$ is R for each nonzero ideal I then R is simple.

Problem 3. Prove that there are infinitely many integers m, n, such that $1 < m < n$, and the least common divisors $(m, n), (m, n+1), (m+1, n)$ and $(m+1, n+1)$ are all greater than $\sqrt{n}/999$.

Problem 4. An $n \times m$ matrix is nice if it contains every integer from 1 to mn exactly once and 1 is the only entry which is the smallest both in its row and in its column. Prove that the number of $n \times m$ nice matrices is $(nm)!n!m!/(n+m-1)!$.

Problem 5. Let $S \subset \mathbb{R}^d$ be a convex compact body with nonempty interior. Show that there is an $\alpha > 0$ such that if $S = \cap_{i \in I} H_i$, where I is an index set and $(H_i)_{i \in I}$ are halfspaces, then there is an $i \in I$ for which $\operatorname{dist}(P, H_i) \geq \alpha \operatorname{dist}(P, S)$.

Problem 6. Let d be a positive integer and $1 < a \leq (d+2)/(d+1)$. For given $x_0, x_1, \ldots, x_d \in (0, a-1)$, let $x_{k+1} = x_k(a - x_{k-d})$, $k \geq d$. Prove that $\lim_{k\to\infty} x_k = a - 1$.

Problem 7. Given a polynomial P, assume that $L = \{z \in \mathbb{C} : |P(z)| = 1\}$ is a Jordan curve. Show that the zeros of P' are in the interior of L.

Problem 8. Let $f : \mathbb{R} \to \mathbb{R}$ be a measurable function such that $f(x+t)-f(x)$ is locally integrable for every t as a function of x. Prove that f is locally integrable.

Problem 9. Does there exist a functional equation[1] that has a solution and the range of any of its solutions is the set of integers?

[1] A *functional equation* has the form $E = 0$, where E is a function form. The set of function forms is the smallest set $\mathcal{F}$ which contains the variables $x_1, x_2, \ldots$, the real numbers $r \in \mathbb{R}$, and for which $E, E_1, E_2 \in \mathcal{F}$ implies $E_1 + E_2 \in \mathcal{F}$, $E_1 \cdot E_2 \in \mathcal{F}$, and $f(E) \in \mathcal{F}$, where f is a fixed function symbol. The solution of the functional equation $E = 0$ is a function $f : \mathbb{R} \to \mathbb{R}$ such that $E = 0$ holds for all values of the variables. E.g. $f\left(x_1 + f(\sqrt{2} \cdot x_2 \cdot x_2)\right) + (-\pi) + (-1) \cdot x_1 \cdot x_1 \cdot x_2 = 0$ is a functional equation.

Problem 10. Let A and B be positive self-adjoint operators on a complex Hilbert space H. Prove that

$$\limsup_{n\to\infty} \|A^n x\|^{1/n} \leq \limsup_{n\to\infty} \|B^n x\|^{1/n}$$

holds for every $x \in H$ if and only if $A^n \leq B^n$ for each positive integer n.

The Miklós Schweitzer Competition is open for all students who currently study in Hungary and also to Hungarian citizens studying abroad who do not hold an MSc (or equivalent) degree in mathematics, computer science or related fields obtained in 2018 or earlier. Those who graduated in 2019 are eligible to participate. Please write your name, affiliation and email address on each sheet, and submit solutions to the problems on separate sheets by 12:00 (noon) CET, 4 November 2019 in one of the following three ways:

- Personally to the secretariat of the Bolyai Institute University of Szeged (6720 Szeged, Aradi vértanúk tere 1.).
- By mail to Péter Kevei, SZTE TTIK Bolyai Intézet, 6720 Szeged, Aradi vértanúk tere 1.
- By e-mail to `kevei@math.u-szeged.hu` in PDF.

the Organizers