Problems posed in the
2016 Miklós Schweitzer Memorial Competition in Mathematics
24 October — 2 November

1. For which complex numbers α does there exist a completely multiplicative, complex valued arithmetic function f such that
$$\sum_{n<x} f(n) = \alpha x + O(1)\,?$$

2. Let $K = (V, E)$ be a finite, simple, complete graph. Let d be a positive integer. Let $\phi\colon E \to \mathbb{R}^d$ be a map from the edge set to Euclidean space, such that the preimage of any point in the range defines a connected graph on the entire vertex set V, and the points assigned to the edges of any triangle in K are collinear. Show that the range of ϕ is contained in a line.

3. Prove that for any polynomial P with real coefficients, and for any positive integer n, there exists a polynomial Q with real coefficients such that $P^2(x) + Q^2(x)$ is divisible by $(1 + x^2)^n$.

4. Prove that there exists a sequence $a(1), a(2), \ldots, a(n), \ldots$ of real numbers such that
$$a(n+m) \le a(n) + a(m) + \frac{n+m}{\log(n+m)}$$
for all integers $m, n \ge 1$, and such that the set $\{a(n)/n : n \ge 1\}$ is everywhere dense on the real line.

   Remark. A theorem of de Bruijn and Erdős states that if the inequality above holds with $f(n+m)$ in place of the last term on the right hand side, where $f(n) \ge 0$ is nondecreasing and $\sum_{n=2}^{\infty} f(n)/n^2 < \infty$, then $a(n)/n$ converges or tends to $(-\infty)$.

5. Does there exist a piecewise linear continuous function $f\colon \mathbb{R} \to \mathbb{R}$ such that for any two-way infinite sequence $a_n \in [0,1]$, $n \in \mathbb{Z}$, there exists an $x \in \mathbb{R}$ with
$$\limsup_{K\to\infty} \frac{\#\{k \le K \,:\, k \in \mathbb{N},\ f^k(x) \in [n, n+1)\}}{K} = a_n$$
for all $n \in \mathbb{Z}$, where $f^k = f \circ \ldots \circ f$ stands for the k-fold iterate of f?

6. Let $\Gamma(s)$ denote Euler's gamma function. Construct an even entire function $F(s)$ that does not vanish everywhere, for which the quotient $F(s)/\Gamma(s)$ is bounded on the right halfplane $\{\mathrm{Re}(s) > 0\}$.

7. Show that the unit sphere bundle of the r-fold direct sum of the tautological (universal) complex line bundle over the space $\mathbb{C}P^\infty$ is homotopically equivalent to $\mathbb{C}P^{r-1}$.

8. For which integers $n > 1$ does there exist a rectangle that can be subdivided into n pairwise noncongruent rectangles similar to the original rectangle?

9. For $p_0, \ldots, p_d \in \mathbb{R}^d$, let

$$S(p_0, \ldots, p_d) = \left\{ \alpha_0 p_0 + \cdots + \alpha_d p_d : \alpha_i \leq 1,\ \sum_{i=0}^{d} \alpha_i = 1 \right\}.$$

Let π be an arbitrary probability distribution on $\mathbb{R}^d$, and choose $p_0, \ldots, p_d$ independently with distribution π. Prove that the expectation of $\pi(S(p_0, \ldots, p_d))$ is at least $1/(d+2)$.

10. Let X and Y be independent, identically distributed random points on the unit sphere in $\mathbb{R}^3$. For which distribution of X will the expectation of the (Euclidean) distance of X and Y be maximal?

Students who do not speak Hungarian are invited to submit solutions in English. These will not be part of the competition, but outstanding solutions will be published in the János Bolyai Mathematical Society's journal *Matematikai Lapok.*

Please solve problems on separate sheets, and submit these no later than 12:00 PM on 2 November 2016, in person to

the János Bolyai Mathematical Society
(1055 Budapest, Falk Miksa u. 12., I.4., door bell 19)

or to

the Secretariat of the Rényi Institute
(1053 Budapest, Reáltanoda u. 13-15.)

or by post to

Frenkel Péter, ELTE TTK Mat. Intézet, Algebra és Számelmélet Tanszék,
1117 Budapest, Pázmány P. stny. 1/c,

or by email in PDF format to

`frenkelp265@gmail.com`

Please write your name on each sheet and your affiliation and email on one sheet.